GRAPHS OF COMMUTATIVELY CLOSED SETS

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Abstract. The present work aims to exploit the interplay between the algebraic properties of rings and the graph-theoretic structures of some associated graphs. We introduce commutatively closed graphs and investigate properties of commutatively closed subsets of a ring with the help of graph theory. In particular, we compute the diameter of matrix rings and artinian semisimple algebras.

Introduction

A subset S of a ring R is *commutatively closed* if for any $a, b \in R$ such that $ab \in S$, we also have $ba \in S$. This notion was introduced and studied in [1]. The aim of this paper is to investigate further this property and, in particular, to involve graph theory in the subject. The diameter of a ring R is defined to be the maximal of the diameter of its commutatively closed classes and gives a measure of the noncommutativity of R. In particular, the diameter of a commutative ring is zero and the diameter of a free algebra on at least two non commuting variables is infinite. It is shown that the set of nilpotent elements is commutatively closed and, in the case of a ring $R = M_n(D)$ of matrices over a division ring D, the diameter of the class of nilpotent matrices is the diameter of R and is shown to be equal to $n-1$. From this, it is easy to compute the diameter of a semisimple algebra. Let us first mention some easy examples. Any subset of a commutative ring is commutatively closed. The set $\{1\}$ is commutatively closed if and only if the ring R is Dedekind finite. Similarly, the set $\{0\}$ is commutatively closed if and only if the ring R is reversible. The intersection of two commutatively closed subsets is easily seen to be commutatively closed and hence every subset of R is contained in a minimal commutatively closed subset: its commutative closure. The *closure* of a subset $S \subset R$ is denoted by $\overline{S}$. In [1] different characterizations of $\overline{S}$ were given. It was shown that this notion is related to various kinds of subsets such as idempotents elements, nilpotent elements, invertible elements, and various kinds of regular elements. It was also used to characterize semicommutative rings, 2-primal rings, and clean rings. We will briefly recall some of the notions used in [1] at the end of this introduction.

In section 1, we describe a graph structure on the sets $\overline{\{a\}}$ and define the diameter. Our aim is to compute the diameter of some classical rings, in particular, the ring of matrices over a division ring. In section 2, we give some ways of constructing $\overline{\{a\}}$ and study some particular rings such as the free algebras. In the last section, we investigate some properties of the commutatively closed graph of matrix rings and semisimple algebras.

2010 *Mathematics Subject Classification.* 16U80,16U70,16S50.

Key words and phrases. commutatively closed sets, graphs, diameter, semisimple algebras.

For two elements $a, b \in R$ we write $a \sim_1 b$ if there exists $c, d \in R$ such that $a = cd$ and $b = dc$. We then define by induction $a \sim_{n+1} b$ if and only if there exists an element $c \in R$ such that $a \sim_1 c$ and $c \sim_n b$. For two elements $a, b \in R$, we also define $a \sim b$ if there exists $n \in \mathbb{N}$ such that $a \sim_n b$. For a subset $S \subseteq R$ we denote S_i the set $\{x \in R \mid x \sim_i s, \text{ for some } s \in S\}$. Since our ring R is unital, the chain S_i is ascending and we have $\overline{S} = \cup_{i \geq 0} S_i$ (for details about these constructions we refer the reader to [1]).

We recall that an element $a \in R$ is called *von Neumann regular* if there is $x \in R$ such that $a = axa$. Similarly, we define $a \in R$ to be a *π-regular* element of R if $a^n x a^n = a^n$ for some $x \in R$ and $n \geq 1$. An element $a \in R$ is called *right (left) π-regular*, if $a^{n+1}x = a^n$ $(xa^{n+1} = a^n)$ for some $x \in R$ and $n \geq 1$. We call $a \in R$ *strongly π-regular* if it is both left and right π-regular.

Throughout this paper R will be a unital ring, $U(R)$ and $N(R)$ will stand for the set of invertible and nilpotent elements of R, respectively. For an element a of a ring R we denote $l(a)$ (resp. $r(a)$) its left (resp. right) annihilator.

Let us mention some results from [1]. They give motivation for the subject and basic information about the results.

Theorem 0.1. [1, Theorem 2.7] *Let $\varphi : R \longrightarrow S$ be a ring homomorphism, then*

(1) *For any $X \subseteq R$, $\varphi(\overline{X}) \subseteq \overline{\varphi(X)}$.*
(2) *If φ is a ring isomorphism, then for any $X \subseteq R$, $\varphi(\overline{X}) = \overline{\varphi(X)}$.*
(3) *If $T \subseteq S$ is commutatively closed in S, then $\varphi^{-1}(T)$ is commutatively closed in R.*
(4) *If S is reversible, $Ker(\varphi)$ is commutatively closed.*
(5) *If S is Dedekind-finite, then $\varphi^{-1}(\{1\})$ is commutatively closed.*

Proposition 0.2. [1, Proposition 3.6]

(1) *If R is Dedekind-finite and $a \in U(R)$, then $\overline{\{a\}} = \{uau^{-1} \mid u \in U(R)\}$.*
(2) *The set of unit $U(R)$ of a ring R is commutatively closed if and only if R is Dedekind-finite.*

Proposition 0.3. [1, Proposition 4.7] *Let k be a commutative field and $n \in \mathbb{N}$, the class of $\overline{\{0\}}$ in $M_n(k)$ is the set of nilpotent matrices.*

For notions related to graph theory, we refer the reader to [7].

1. Commutatively closed graphs and their diameters

We start this section with some examples of commutatively closed sets.

Examples 1.1. (1) A ring R is reversible if and only $\{0\}$ is commutatively closed;

(2) A ring R is Dedekind finite if and only if $\{1\}$ is commutatively closed;
(3) For a right R-module M_R the set $E = \{u \in R \mid ann_M(u-1) \neq 0\}$ is commutatively closed. Indeed, if $u = ab \in E$ and $0 \neq m \in M$ is such that $m(ab-1) = 0$, then $ma \neq 0$ and $ma(ba-1) = m(ab-1)a = 0$. This gives that $ba \in E$.
(4) The ring R is symmetric if and only if for every $a, b, c \in R$, $abc = 0$ implies that $acb = 0$. This can be translated into asking that for any $a \in R$, $r(a) := \{b \in R \mid ab = 0\}$ is commutatively closed. We refer the reader to [3] for more information on this kind of ring.

(5) $U(R)-1$ is always commutatively closed. More generally, denoting the set of regular elements by $Reg(R)=\{r\in R \mid \exists x\in R$ such that $r=rxr\}$ we have that $Reg(R)-1$ is always commutatively closed. A similar result is true for the set of unit regular elements and also for the set of strongly π-regular elements. For proofs of these facts we refer the reader to [5].
(6) We consider $Z_l(R)=\{a\in R \mid r(a)\neq 0\}$ the set of left zero divisors. We define similarly $Z_r(R)$ the set of right zero divisors. Similarly, as in the item above, $Z_r(R)+1$ (resp. $Z_l(R)+1$) is commutatively closed ([1]).
(7) The set $N(R)$ of nilpotent elements of a ring R is easily seen to be commutatively closed.
(8) Recall that an element $r\in R$ is strongly clean if there exist an invertible element $u\in U(R)$ and an idempotent $e^2=e\in R$ such that $eu=ue$ and $r=e+u$. It is proved in Theorem 2.5 of [3] that the set of strongly clean elements is commutatively closed. In the same paper, the author also shows that the set of Drazin (resp. almost, pseudo) invertible elements is also commutatively closed.

The first three statements of the following results were obtained in [1], so we will only prove the last one.

Proposition 1.2. (1) *For any $n\geq 1$ and $a,b\in R$, we have $a\sim_n b$ if and only if there exist two sequences of elements in R $x_1,x_2,\dots,x_n$ and $y_1,y_2,\dots,y_n$ such that $a=x_1y_1, y_1x_1=x_2y_2, y_2x_2=x_3y_3,\dots,y_nx_n=b$.*
(2) *If $a\sim_n b$, then $a-b$ is a sum of n additive commutators.*
(3) *If $a\sim_n b$, then there exist $x,y\in R$ such that $ax=xb$ and $ya=by$. Moreover, for $l\in\mathbb{N}$, we also have $b^{n+l}=ya^lx$ and $a^{n+l}=xb^ly$. In particular, $b^n=yx$ and $a^n=xy$.*
(4) *If elements a,b in a ring R are such that $a\sim_m b$, then for any $0<l<m$, we have $a^l\sim_q b^l$, where q is such that $q=\lceil \frac{m}{l}\rceil$*

Proof. (4) We can write $m=lq-r$ for some $0\leq r<l$ and, since $1_R\in R$, $a\sim_m b$ implies that we also have $a\sim_{m+r=lq} b$. We prove that $a^l\sim_q b^l$ by induction on q. If $q=1$, the statement (3) above shows that $a^m=xy$ and $b^m=yx$ for some $x,y\in R$ and hence $a^m\sim_1 b^m$.

So let us now assume that $q>1$ and put $n:=m+r=lq$, so that we have $a\sim_n b$. We use the same notations as in statement (1) and assume that there exist two sequences of elements in R $x_1,x_2,\dots,x_n$ and $y_1,y_2,\dots,y_n$ such that $a=x_1y_1, y_1x_1=x_2y_2, y_2x_2=x_3y_3,\dots,y_nx_n=b$. We than have that $a=x_1y_1\sim_l y_lx_l\sim_{l(q-1)} b$. The case $q=1$ and the induction hypothesis lead to $a^l\sim_1 (y_lx_l)^l\sim_{q-1} b^l$. This gives the conclusion. □

Remarks 1.3. (1) With the same notations as in the above proposition 1.2, if $a\sim_n b$ and x,y are such that $ax=xb$ and $by=ya$, then left multiplication by the elements y and x give rise to maps in $Hom_R(R/aR,R/bR)$ and $Hom_R(R/bR,R/aR)$. We then have the compositions $L_x\circ L_y=L_{a^n}$ and $L_y\circ L_x=L_{b^n}$. In particular, if $a\sim b$ then $Hom_R(R/aR,R/bR)\neq\{0\}$.
(2) In section 3, we will study the ring of matrices over a division ring D. It is interesting to observe that if $A,B\in M_n(D)$ are such that $A\sim B$, then the central spectrum $Sp(A)$ and $Sp(B)$ are equal. Indeed, there exists $m\in\mathbb{N}$ such that $A\sim_m B$. Proposition 1.2 (3) shows that there exist

matrices $X, Y \in M_n(D)$ such that $AX = XB$ and $YA = BY$. So, if $Av = \lambda v$ for some $0 \neq v \in D^n$ and $\lambda \in k$ (k is the center of D), then $BYv = YAv = Y\lambda v = \lambda Yv$. Noting that $\lambda^m v = A^m v = XYv$, we conclude that $Yv \neq 0$ and hence λ is indeed a central eigenvalue of B. Let us remark that in the case D is commutative, the equality between the spectrums is a consequence of the fact that the characteristic polynomials of A and B are the same (cf. [1]).

In the following, we define the commutatively closed graph and show that the graph $C(a)$ is always connected for every $a \in R$. Also, we analyze some of its other properties.

Definitions 1.4. (1) *Let a be an element in a ring R and let $C(a)$ denote its commutative closure as defined in the introduction. We define a graph structure with the elements of $C(a) = \overline{\{a\}}$ as vertices and two distinct vertices x and y of $C(a)$ are said to be adjacent if and only if $y \in \{x\}_1$. The commutatively closed graph of a ring R is the union of all the graphs $C(a)$, for $a \in R$. It will be denoted by $C(R)$.*

(2) *Let a be an element in R. In the class $\overline{\{a\}}$ of a, we define a distance as follows: For two elements $x, y \in \overline{\{a\}}$, we put $d(x, y) = min\{n \in \mathbb{N} \mid y \sim_n x\}$.*
One can easily check that d is indeed a distance defined on $\overline{\{a\}}$.

(3) *Let R be a ring and $a \in R$, the diameter of a graph $C(a)$ is defined as follows:*
$$diam\big(C(a)\big) = sup\big\{d(x, y) \mid x, y \in \overline{\{a\}}\big\}.$$
Also, we define the diameter of a set $S \subseteq R$ of a ring S as follows:
$$diam(S) = sup\big\{diam\big(C(a)\big) \mid a \in S\big\}.$$

Theorem 1.5. (1) *For $a, b \in R$, we have $a \sim b$ if and only if $b \in \overline{\{a\}}$.*
(2) *The relation $\sim$ on R is an equivalence relation.*
(3) *If $b \in C(a)$ then, for any $l \in \mathbb{N}$, $b^l \in C(a^l)$.*
(4) *For $a \in R$, the graph $C(a)$ is connected.*
(5) *A subset S of R is closed and connected if and only if it is the closure of an element of R.*
(6) *If $\overline{S}$ is the closure of a subset $S \subseteq R$ then $diam(S) = diam(\overline{S})$.*

Proof. We leave the easy proof to the reader. □

Proposition 1.6. *Let a be an element in a ring R. If $n \in \mathbb{N}$ is the smallest integer such that $\overline{a} = \{a\}_n$, then $n \leq diam_R\big(C(a)\big) \leq 2n$.*

Proof. We know that the distance between a and every other element of $\{a\}_n$ is at most n. Now, if $b, c \in C(a)$ then $d(b, c) \leq d(b, a) + d(a, c) \leq 2n$. Then $diam_R\big(C(a)\big) \leq 2n$. The fact that n is minimal such that $\overline{a} = \{a\}_n$ implies that $n \leq diam_R\big(C(a)\big)$. □

Lemma 1.7. *Let R and S be two rings, and $(a, b) \in R \times S$. Then $\overline{\{(a, b)\}} = \overline{\{a\}} \times \overline{\{b\}}$.*

Proof. Let $(c, d) \in \overline{\{(a, b)\}}$. Thus $(c, d) \sim_n (a, b)$, for some $n \geq 0$. So there are $(c_1, d_1), \ldots, (c_n, d_n) \in R \times S$ such that $(c, d) \sim_1 (c_1, d_1) \sim_1 \cdots \sim_1 (c_n, d_n) = (a, b)$. One can easily see that $c \sim_n a$ and $d \sim_n b$. Hence $(c, d) \in \overline{\{a\}} \times \overline{\{b\}}$.

Now, let $(c,d) \in \overline{\{a\}} \times \overline{\{b\}}$. Since $c \in \overline{\{a\}}$ and $d \in \overline{\{b\}}$, thus there are $m, n \in \mathbb{N}$ such that $c \sim_m a$ and $d \sim_n b$. So we have $c \sim_1 c_1 \sim_1 \cdots \sim_1 c_m = a$ and $d \sim_1 d_1 \sim_1 \cdots \sim_1 d_n = b$, for some $c_1, \ldots, c_m \in R$ and $d_1, \ldots, d_n \in S$. We may assume that $m < n$. We have $(c,d) \sim_1 (c_1, d_1) \sim_1 \cdots \sim_1 (c_m, d_m) = (a, d_m) \sim_1 (a, d_{m+1}) \sim_1 \cdots \sim_1 (a, d_n) = (a,b)$. Hence $(c,d) \sim_n (a,b)$, and so $(c,d) \in \overline{\{(a,b)\}}$. □

Proposition 1.8. *Let R, S be two rings, and let $diam(R) = d_1$ and $diam(S) = d_2$. Then $diam(R \times S) = max\{d_1, d_2\}$.*

Proof. Assume that $(a,b) \in R \times S$. For every $(a_1, b_1), (a_2, b_2) \in \overline{\{(a,b)\}}$, we have $a_1, a_2 \in \overline{\{a\}}$ and $b_1, b_2 \in \overline{\{b\}}$, by Lemma 1.7. Since $diam(R) = d_1$ and $diam(S) = d_2$, then $d(a_1, a_2) \leq d_1$ and $d(b_1, b_2) \leq d_2$. Thus there exist $t \leq d_1$ and $s \leq d_2$ such that $a_1 \sim_t a_2$ and $b_1 \sim_s b_2$. Then we have $a_1 \sim_1 c_1 \sim_1 \cdots \sim_1 c_t = a_2$ and $b_1 \sim_1 v_1 \sim_1 \cdots \sim_1 v_s = b_2$, for some $c_1, \ldots, c_t \in R$ and $v_1, \ldots, v_s \in S$. Let $t < s$. We have $(a_1, b_1) \sim_1 (c_1, v_1) \sim_1 \cdots \sim_1 (c_t, v_t) = (a_2, v_t) \sim_1 (a_2, v_{t+1}) \sim_1 \cdots \sim_1 (a_2, v_s) = (a_2, b_2)$. Hence $d\big((a_1, b_1), (a_2, b_2)\big) \leq max\{d_1, d_2\}$. Therefore $diam(R \times S) = max\{d_1, d_2\}$. □

Remark 1.9. In the above proposition, we showed that if R and S are two rings with finite diameter, then the diameter of $R \times S$ is also finite.

Let us remark that is easy to construct elements a, b, c, d in a ring such that $a \sim_1 b$ and $c \sim_1 d$, but there is no path between ac and bd. For instance, consider the free algebra $K < X, Y >$ where K is a field. Let $a = XY, b = YX, c = d = X^2Y$. We leave to the reader to check that there is no path from $ac = XYX^2Y$ to $bd = YX^3Y$.

Lemma 1.10. *If R is a Dedekind finite ring and $a \in U(R)$, then $diam_R(C(a)) = 1$.*

Proof. Since R is Dedekind finite and $a \in U(R)$, we have $\overline{\{a\}} = \{uau^{-1} \mid u \in U(R)\}$, by Proposition 0.2. So for every $b, c \in \overline{\{a\}}$, there exist $u, v \in U(R)$ such that $b = uau^{-1}$ and $c = vav^{-1}$. Hence $b = uv^{-1}cvu^{-1}$. Thus b and c are adjacent. Therefore $diam_R(C(a)) = 1$. □

Proposition 1.11. *1) R is commutative if and only if $diam(R) = 0$.*
2) Let R be a division ring. Then $diam(R) = 1$.

Proof. 1) The first statement is a direct consequence of the definition.
2) The second statement is easily obtained from Lemma 1.10. □

Theorem 1.12. *If R is not Dedekind-finite, then $diam(R) = \infty$.*

Proof. Assume that R is not Dedekind-finite. Thus there exist $a, b \in R$ such that $ab = 1$ but $ba \neq 1$. So we have nonzero element $e_{ij} = b^i(1 - ba)a^j$. Consider $A = e_{12} + e_{23} + \cdots + e_{n-1,n}$. One can easily check that $A^n = 0$ and $A^{n-1} = e_{1n} \neq 0$, since $ba \neq 1$. Write $A = (e_{11} + e_{22} + \ldots e_{n-1,n-1})A \sim_1 A(e_{11} + e_{22} + \ldots e_{n-1,n-1}) = e_{12} + e_{23} + \cdots + e_{n-2,n-1}$. Continuing this process, we conclude $A \in \{0\}_{n-1}$. We claim that $A \notin \{0\}_{n-k}$ for $1 < k < n$. Let $A \in \{0\}_{n-k}$. Thus $A \sim_{n-k} 0$. By Proposition 1.2 (3), there exist $X, Y \in R$ such that $A^{(n-k)+l} = X(0)^lY$ $(\forall l \in \mathbb{N})$. If we put $l = k+1$, then $A^{n-1} = 0$, which is a contradiction. Hence $d(0, A) = n - 1$ for every $n \in \mathbb{N}$. Therefore $diam(R) = \infty$. □

2. Constructing the closure of a subset of R

Let us first remark that the commutative closure $\overline{S}$ of a subset $S \subseteq R$ is the union of the commutative closure $\overline{\{s\}}$ of its elements $s \in S$. On the other hand to construct the commutative closure of an element we need to use factorizations of this element. We will use the following tools to build elements of $\overline{s}$, for $s \in R$.

- For any $a \in R$, we have $a \sim_1 a(1+b)$ (resp. $a \sim_1 (1+b)a$) for any $b \in l(a)$ (resp. $b \in r(a)$).
- Somewhat more general than the above point, let us remark that if $xb = 0$ we always have $xy \sim_1 (y+b)x$ (resp. if $c, y \in R$ are such that $cy = 0$ then $xy \sim_1 y(x+c)$). We even have, for $a = xy^n$, that $(y+r(x))^n x \in \{a\}_n$ (and for $b = x^n y$ we have $y(x+l(y))^n \in \{b\}_n$).
- For $a, b \in R$ we have $a(1+ba) \sim_1 a(1+ab)$.

We leave the short proofs of these statements to the reader and start to apply them to different cases.

Example 2.1. Let $R = M_2(k)$ and $A, B \in R$. Also let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. Thus $AB = 0$ and $BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Since $1 + AB = I$ and $1 + BA = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, we have $1 + BA \notin \overline{\{1+AB\}} = \overline{\{I\}} = \{I\}$, as desired.

Lemma 2.2. *Let R be any ring, and let $a, b \in R$ and $u \in U(R)$ such that $b = uau^{-1}$. Then, for any $n \geq 1$, $\{a\}_n = \{b\}_n$.*

Proof. Let c be any element in $\{b\}_n$. According to Proposition 1.2, there are two sequence of elements $x_1, \dots, x_n, y_1, \dots, y_n$ in R such that $b = x_1y_1$, $y_1x_1 = x_2y_2, \dots, y_nx_n = c$. We thus have $a = u^{-1}bu = (u^{-1}x_1)(y_1u) \sim_1 y_1x_1 \sim_{n-1} c$. We thus conclude that $a \sim_n c$. This yields the result. $\square$

Example 2.3. Let $R = k\langle x, y, z, t\rangle/I$, where k is a field and I is the ideal generated by the element $xy - zt$. Then $yx, tz \in \{xy\}_1$ and $d(yx, tz) = 2$.

Theorem 2.4. *Let k be a field and $R = K\langle x, y\rangle$ be the free k-algebra. Then $diam(R)$ is infinite.*

Proof. Consider $x, y \in R$. We show that for every $l \in \mathbb{N}$, $d(x + yx^l, x + x^ly) = l$. Since $x + yx^l = (1 + yx^{l-1})x \sim_1 x(1 + yx^{l-1}) = (1 + xyx^{l-2})x \sim_1 x(1 + xyx^{l-2}) = (1 + x^2yx^{l-3})x \sim_1 \cdots \sim_1 x(1 + x^{l-1}y) = x + x^ly$, so $d(x + xyx^l, x(1 + x^ly)) \leq l$.

The path we just described between the two elements $x + xyx^l$ and $x + x^ly$ is the only path between these two elements. This is a consequence of the fact that, for $r, s \geq 1$, the only factorization in R of $x + x^syx^r$ are $x + x^syx^r = x(1 + x^{s-1}yx^r) = (1 + x^syx^{r-1})x$. We leave the arguments to the reader. $\square$

Example 2.5. Let R be a ring and I a commutatively closed ideal of R. Also, let $diam\left(\frac{R}{I}\right)$ is finite. Then the diameter $diam(R)$ is not necessarily finite. Consider $R = k[x_1, x_2, \dots, x_m, \dots][x; \sigma]$, where k is field and σ is an automorphism on $k[x_1, x_2, \dots, x_m, \dots]$ such that $\sigma(x_i) = x_{i+1}$. Since the chain $x_1x \sim_1 xx_1 =$

$x_2x \sim_1 xx_2 = x_3x \sim_1 \ldots$ is infinite we get that $diam(R)$ is infinite. Also assume that $I = (x)$. We can easily see that I is commutatively closed. Since $\dfrac{R}{I} \cong k[x_1, x_2, \ldots, x_m, \ldots]$ is commutative, we have $diam(k[x_1, x_2, \ldots, x_m, \ldots]) = 0$.

Example 2.6. Let R and S be two rings and $\varphi : R \to S$ a morphism. In general we can get any relation between $diam(R)$ and $diam(S)$. For instance, there is a homomorphism $\varphi : k\langle x, y\rangle \to k$, where k is a field. We know $diam(k) = 1$ while $diam(k\langle x, y\rangle)$ is infinite.

The next proposition establishes a nice connection of our study with the notion of stably equivalent elements in a ring. In fact, we will just use a very special case of this notion and hence we do not introduce a formal definition (see PM Cohn [2] for more information). Let us recall that two square matrices $A, B \in M_n(R)$ are said to be *equivalent* if there exist invertible matrices P, Q such that $PAQ = B$. We shall say that A and B are *stably equivalent* if $diag(A, I)$ is equivalent to $diag(B, J)$ for some unit matrices I, J (not necesserily of same size).

Proposition 2.7. *Let x, y be elements in a ring R. The 2×2 diagonal matrices $diag(1 - xy, 1)$ and $diag(1 - yx, 1)$ are equivalent.*

Proof. First remark that

$$\begin{pmatrix} y & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y & yx-1 \\ 1 & x \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} x & 1 \\ yx-1 & y \end{pmatrix},$$

so that the matrices on the right-hand side of the above equalities are invertible. We have:

$$\begin{pmatrix} y & yx-1 \\ 1 & x \end{pmatrix}\begin{pmatrix} 1-xy & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & 1 \\ yx-1 & y \end{pmatrix} = \begin{pmatrix} 1-yx & 0 \\ 0 & 1 \end{pmatrix}.$$

This proves our proposition. □

Remark 2.8. Let us remark that the above result is due to the fact that, in the language used in [2], the equality $(1 - xy)x = x(1 - yx)$ is comaximal.

This leads to the following statement:

Proposition 2.9. *Let S be a connected subset of a ring R. Then S is commutatively closed if and only if for any two elements $a, b \in S$ we have that the diagonal matrices $diag(1 - a, 1)$ and $diag(1 - b, 1)$ are equivalent in $M_2(R)$.*

Proof. If S is commutatively closed and connected (equivalently $S = C(a)$ for some $a \in S$) subset of R and $a, b \in S$, then there is a path from a to b in S and, as in Proposition 1.2, we have two sequences $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ such that $a = x_1y_1, y_1x_1 = x_2y_2, y_2x_2 = x_3y_3, \ldots, y_nx_n = b$ and Proposition 2.7 easily implies that $diag(1 - a, 1)$ and $diag(1 - b, 1)$ are equivalent.

Conversely, if $a = xy \in S$, then Proposition 2.7 gives that the diagonal matrices $diag(1 - xy, 1)$ and $diag(1 - yx, 1)$ are equivalent and hence $yx \in S$. □

In the next corollary, we get some generalizations of classical examples.

Corollary 2.10. *The set $1-S$ is commutatively closed for any one of the following subsets S of R:*

1. *$S = U_r(R)$ (resp. $S = U_l(R)$ or $S = U(R)$), the set of right (rep. left, two sided) invertible elements of R.*

(2) $S = \{A \in M_n(K) \mid Rank(A) = l\}$, *where K is a field and $l \leq n \in \mathbb{N}$.*
(3) $S = reg(R)$ *the set of regular elements of R.*
(4) *S is the set of strongly π-regular elements.*
(5) *S is the set of left (right) zero divisors in R.*

Proof. We refer the reader to [1] for the proofs or references for these statements. $\square$

3. On Commutatively closed graph over matrix rings

In this section, we study some properties of the commutatively closed graph over matrix rings.

For a ring R and $n \in \mathbb{N}$, we denote $N_n(R)$ the set of elements of R that are nilpotent of index n.

Proposition 3.1. *Let R be a ring. Then*
(1) *For any $i \in \mathbb{N}$, we have $\{0\}_i \subseteq N_{i+1}(R)$. In particular, $\overline{0} \subseteq N(R)$.*
(2) *For any strictly upper triangular matrix $U \in M_n(R)$, $U \in \{0\}_{n-1} \subseteq \overline{\{0\}}$.*
(3) *Let $U_n(R) \subseteq M_n(R)$ be the set of all $n \times n$ strictly upper triangular matrix over R. Then $diam\big(U_n(R)\big) \leq 2(n-1)$.*

Proof. (1)This is easily proved by induction using Proposition 1.2.

(2) We may assume that $U \neq 0$ and we denote the lines of U by $L_1, L_2, \ldots, L_n$. In fact, the last line L_n is zero, and we define $r \in \{1, \ldots, n-1\}$ to be minimal such that L_i is zero for $i > r$. We will prove that $U \in \{0\}_r$ by induction on r. We write

$$U = \begin{pmatrix} I_{r,r} & 0 \\ 0 & 0 \end{pmatrix} U \quad \text{and} \quad B := U \begin{pmatrix} I_{r,r} & 0 \\ 0 & 0 \end{pmatrix} \in \{U\}_1,$$

where $I_{r,r}$ denotes the identity matrix of size $r \times r$.
If $r = 1$, we get that $B = 0 \in M_n(D)$ and this yields the thesis.
If $r > 1$, write $B = (R_1, \ldots, R_n)$ where R_i is the i^{th} row of B. The matrix B is easily seen to be upper triangular and such that the rows $R_r, \ldots, R_n$ are zero. This means that this matrix has at least one more zero row than the matrix U. The induction hypothesis gives that $B \in \{0\}_{r-1}$, but then $U \in \{B\}_1 \subseteq \{0\}_r \subseteq \overline{\{0\}}$, as required.
(3) By the above statement (2), we know $U_n(R) \subseteq \{0\}_{n-1} \subseteq \overline{\{0\}}$. So that for two matrices $A, B \in U_n(R)$, we have $A \sim_{n-1} 0 \sim_{n-1} B$. This yields the conclusion. $\square$

We will now determine the diameter of the class $C(0) \in M_n(D)$ where D is a division ring. The following lemma is well known, we give a proof for completeness.

Lemma 3.2. *Every nilpotent matrix with coefficients in a division ring is similar to a strictly upper triangular matrix.*

Proof. The proof is based on the fact that any nonzero column can be the first column of an invertible matrix. So let $A \in M_n(D)$ be a nilpotent matrix with coefficients in a division ring D and let $u \in M_{n,1}(D)$ be a nonzero column such that $Au = 0$. Let $U \in M_n(D)$ be an invertible matrix having u as its first column. We conclude that

$$U^{-1}AU = \begin{pmatrix} 0 & r \\ 0 & A_1 \end{pmatrix},$$

for some row $r \in M_{1,n-1}(D)$. It is easy to check that A_1 is again nilpotent. An easy induction on the size of the nilpotent matrix yields the proof. $\square$

The next result was proved in [1] for matrices with coefficients over fields.

Proposition 3.3. *Let D be a division ring and $n \in \mathbb{N}$, the class of $\overline{\{0\}}$ in $R = M_n(D)$ is the set of nilpotent matrices.*

Proof. We have seen that, in any ring, $\{0\}_i \subseteq N(R)_{i+1}$ (cf. Proposition 3.1). Conversely, if $A \in M_n(D)$ is nilpotent, the above lemma 3.2 shows that there exists an invertible matrix P and a strictly upper triangular matrix $U \in M_n(D)$ such that $PAP^{-1} = U$. Since the class of an element is the same as the class of any of its conjugate, we conclude that $\overline{\{A\}} = \overline{\{U\}}$. Proposition 3.1 (1) and (2) implies that $\overline{\{U\}} = \overline{\{0\}}$. $\square$

A Jordan block J_l (associated to zero) is a matrix of the form

$$J_l = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & 1 \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{pmatrix} \in M_l(D) \qquad (1)$$

If $A \in M_n(D)$ is a nilpotent matrix, where D is a division ring, A is similar to a diagonal sum of Jordan blocks. This is classical if D is commutative and for a proof in a noncommutative setting we may refer to Chapter 8 of P.M. Cohn's book ([2]) or to the more recent paper [4]. Let us notice that $J_1 = 0$.

Lemma 3.4. *Let J_l be a matrix block of size $l > 1$. Then $J_l \sim_{l-1} 0$*

Proof. As above, we write J_l for the Jordan matrix presented in (1).

We proceed by induction on l. If $l = 2$, we have

$$J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \sim_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Similarly, for $l > 2$, we have

$$J_l = (\sum_{i=1}^{l-1} e_{ii}) J_l \sim_1 J_l (\sum_{i=1}^{l-1} e_{ii}) = \begin{pmatrix} J_{l-1} & 0 \\ 0 & 0 \end{pmatrix} \sim_{l-2} 0.$$

Where we have used the induction hypothesis: $J_{l-1} \sim_{l-2} 0$. This implies that $J_l \sim_{l-1} 0$, as required. $\square$

Let us extract from the above proof the following observation.

Corollary 3.5. *If $l > 1$, we have $J_l \sim_1 diag(J_{l-1}, J_1)$.*

We will now look more closely to the class of nilpotent matrices.

Proposition 3.6. *Let D be a division ring. Then for $A \in M_n(D)$, we have*

$$A^{l+1} = 0 \Leftrightarrow A \sim_l 0.$$

Proof. We suppose that $A \neq 0$. We know (cf. Proposition 3.3) that the class $C(0)$ consists of all the nilpotent matrices. Thanks to Lemma 2.2 we know that we can replace A by a conjugate to evaluate the length of path from A to 0. Using a result from [4], we know that there exists an invertible matrix $P \in GL_n(D)$ such that PAP^{-1} is of the form $diag(J_{n_1}, J_{n_2}, \dots, J_{n_s})$ where the square matrices $J_i \in M_i(D)$

are of the form given in (1) above. Moreover the maximal size of the Jordan blocks is $l+1$ i.e., for all $1 \leq i \leq s$, we have $n_i \leq l+1$. Since for any integer $i \in \mathbb{N}$, we have $\{0\}_i \subseteq \{0\}_{i+1}$, the above Lemma 3.4 implies that, for all $1 \leq i \leq s$, $J_{n_i} \sim_l 0$. This easily leads to the conclusion that $A \sim_l 0$.
The converse was proved in Lemma 3.1. □

Theorem 3.7. *Let $R = M_n(D)$, where D is a division ring. Then, for nilpotent matrices $A, B \in M_n(k)$, with nilpotent indexes $n(A), n(B)$ respectively, we have $d(A,B) \leq max\{n(A), n(B)\} - 1$. In particular, $diam_R\big(C(0)\big) = n-1$.*

Proof. We know that the class of zero in $M_n(D)$ is exactly the set of nilpotent matrices. Hence the matrices $A, B \in C(0)$ and the distance between A to B is the shortest path from A, B in the graph defined by $C(0)$. Let us write $l = n(A)$ and $s = n(B)$, by symmetry we may assume that $l \geq s$. Let $diag(J_{n_1}, J_{n_2}, \ldots, J_{n_r})$ be the Jordan form of B, where $r \geq 1$ and $s = n_1 \geq n_2, \cdots \geq n_r$. We will use induction on r. In the proof, to avoid heavy notations, we will write, $(c_1, c_2, \ldots, c_t)$ for $diag(c_1, c_2, \ldots, c_t)$ (where $c_1, c_2, \ldots, c_t$ are square matrices).
If $r = 1$, then $B = J_s$ and $A = (J_l, A')$ where A' is a nilpotent matrix of index $\leq l$. Using repeatedly Corollary 3.5, we can write $A = (J_l, A') \sim_{l-s} (J_s, A'')$ and $(A'')^s = 0$, so $d(A, (J_s, A'')) \leq l - s$ and

$$d(A,B) \leq d(A, (J_s, A'')) + d((J_s, A''), B).$$

Using Proposition 3.6, we get $d((J_s, A''), B) = d((J_s, A''), J_s) = d(A'', 0) \leq s - 1$ and we conclude $d(A, B) \leq l - 1$, as required.

Suppose now that the formula is proved for matrices B having less than $r > 1$ Jordan blocks and consider a matrix $B = (J_{n_1}, \ldots, J_{n_r}) = (J_s, B')$, with $(B')^s = 0$. As above we have $A = (J_l, A') \sim_{l-s} (J_s, A'')$ and the induction hypothesis gives also $d(A'', B') \leq max\{n(A''), n(B')\} - 1$. This gives $d(A, B) \leq d(A, (J_s, A'')) + d((J_s, A''), B) \leq l - s + d(A'', B') \leq l - s + max\{n(A''), n(B')\} - 1 \leq l - s + s - 1 = l - 1$, as required.

In particular, since the maximal index of nilpotency for matrices in $R = M_n(k)$ is n, and $d(J_n, 0) = n - 1$ we get that $diam_R C(0) = n - 1$. □

We will now show that the diameter of the matrix ring $M_n(D)$, over a division ring D, is itself $n - 1$. The following lemma is far from surprising but needs to be proved.

Lemma 3.8. *Suppose that D is a division ring and that $A, B \in M_n(D)$ are of the form*

$$A = \begin{pmatrix} U & 0 \\ 0 & N \end{pmatrix}, B = \begin{pmatrix} V & 0 \\ 0 & M \end{pmatrix}$$

where $U \in GL_r(D)$ and $V \in Gl_s(D)$ are invertible matrices and N, M are nilpotent matrices. If $A \sim B$ then $r = s$.

Proof. Let us first remark that there exists $l \in \mathbb{N}$ such that $N^l = 0$ and $M^l = 0$. Theorem 1.5 implies that we may assume $M = 0$ and $N = 0$. Taking powers again, Proposition 1.2 shows that we may assume $A \sim_1 B$. So let us write $A = XY, B = YX \in M_n(R)$ and decompose X and Y as follows

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}, \quad \text{and} \quad Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}$$

where $X_1 \in M_{r\times s}(D)$, $Y_1 \in M_{s\times r}(D)$. This fixes the size of all the other matrices appearing in X and Y. Since M and N are zero, the two equations $AX = XB$ and $YB = BY$ quickly imply that X_2, X_3, Y_2, Y_3 are all zero, and we get $U = X_1Y_1$ and $V = Y_1X_1$. Since U and V are invertible we easily conclude that $r = s$. □

Theorem 3.9. *Let D be a division ring, $n \in \mathbb{N}$, $n \geq 2$. Then*

$$diam(M_n(D)) = n - 1.$$

Proof. A consequence of the Fitting Lemma is that any matrix $A \in M_n(D)$ is similar to a block diagonal matrix of the form $diag(U, N)$ where U is an invertible matrix and N a nilpotent matrix. Lemma 2.2 shows that to compute the distance between two different matrices that are in the same commutative class, we may use similar matrices. Thus we need to compute $d(A, B)$ where A and B are of the form $A = diag(U, N)$ and $B = diag(V, M)$. The preceding lemma 3.8 shows that $U, V \in Gl_s(D)$ and $N, M \in M_{n-s}(D)$. Theorem 3.7 and Lemma 1.10 show that we may assume $0 < s < n$.

Since the matrices N and M are nilpotent we conclude that their distance is less than or equal to $n - 2$. This means that there is a sequence of factorizations of length $\leq n-2$ linking M and N. We claim that the matrices U and V are in fact similar. Assume that $A \sim_r B$. According to Proposition 1.2 we know that there exist matrices $X, Y \in M_n(D)$ such that $AX = XB$ and $YA = BY$. Moreover for any $l \in \mathbb{N}$, we have that $A^{r+l} = XB^lY$. Choosing l such that $M^l = N^l = 0$, and writing X, Y as blocks matrices with $X_1, Y_1 \in M_s(D)$, this last equality shows that

$$\begin{pmatrix} U^{r+l} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \begin{pmatrix} V^l & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}.$$

Comparing the blocks on the top left corner gives $U^{r+l} = X_1V^lY_1$. Since U is invertible, we conclude that the matrices X_1 and Y_1 are also invertible. Now, comparing again the top left corner blocks of the equality $AX = XB$, we get $UX_1 = X_1V$. This shows that $U$and V are similar, as claimed. This implies that $U \sim_1 V$. Since $A = diag(U, N)$ and $B = diag(V, M)$ (with sizes of U and N greater or equal to 1). We conclude, thanks to Theorem 3.7, that $d(A, B) = d(M, N) \leq n - 2$. Hence the class with the biggest distance is the class of nilpotent matrices, so that Theorem 3.7 implies that $Diam(M_n(D)) = n - 1$. □

Theorem 3.10. *Let R be a non reduced semisimple ring and $R = M_{n_1}(D_1) \times \cdots \times M_{n_l}(D_l)$ be its Wedderrburn Artin decomposition where $D_1, \ldots, D_l$ are division rings. Then*

$$diam(R) = \max\{n_i - 1 \mid 1 \leq i \leq l\}.$$

Proof. This is a simple consequence of Theorem 3.9 and Proposition 1.8 □

Definition 3.11. *The girth of a graph G, denoted by $gr(G)$, is the length of shortest cycle in G, provided G contains a cycle; otherwise $gr(G) = \infty$.*

Note that if R is a ring, then we define the commutatively closed girth of R as follows:

$$gr(C(R)) = min\{gr\big(C(a)\big) \mid a \in R\}.$$

Theorem 3.12. *Let D be a division ring and $n \in \mathbb{N}$ with $n \geq 2$. Then $gr_{M_n(D)}\big(C(0)\big) = 3$.*

Proof. We know the class of $\overline{\{0\}}$ in $R = M_n(D)$ is the set of nilpotent matrices, where D is a division ring (cf. Proposition 3.3). It is enough, to find three nilpotent matrices that form a cycle. It is easy to see that $E_{1n}, E_{n1} \in \{0\}_1$ (Since $E_{1n} = E_{11}(E_{1n})$, $0 = E_{1n}(E_{11})$, $E_{1n} = E_{1n}E_{nn}$ and $E_{nn}E_{1n} = 0$) and $E_{1n} \sim_1 E_{n1}$. Hence $E_{1,n}$, $E_{2,n-1}$ and 0 form a cycle. Therefore $gr\big(C(0)\big) = 3$. □

As an immediate consequence of Theorem 3.12 and definition of $gr(C(R))$, we have the following.

Corollary 3.13. *Let D be a division ring $n \in \mathbb{N}$ with $n \geq 2$. Then $gr\big(C(M_n(D))\big) = 3$.*

Corollary 3.14. *If R is a non reduced semisimple ring then $gr\big(C(R)\big) = 3$.*

Acknowledgment

This work was partially written while the first author visited the Artois University. She would like to thank the welcome she received from the members of this institution. She also thanks the ministry of science in Iran, and Shahrood University of technology for their financial support during this stay.

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